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ON THREE-DIMENSIONAL EFFECTS NEAR THE VERTEX OF A CRACK IN A THIN PLATE*

S.A. NAZAROV

The influence of the face profile of a crack on the stress-strain state of a thin plate is discussed. A three-dimensional boundary layer appears in the immediate neighbourhood of the crack ends, while linear combinations of special singular solutions of the generalized plante-stress state serve as asymptotic correction terms far from the ends of the crack. The energy-balance formula for crack propagation is sharpened. Formulations of two-dimensional problems which include two asymptotic terms are indicated, (and in particular, the idea of an effective plane-projected crack length is introduced).

1. Statement of the problem and preliminaries. Let G_0 be a domain in the plane $\mathbf{R}^2$, bounded by a simple smooth closed contour Γ and containing an interval $M = \{y \in \mathbf{R}^2\colon y_2 = 0, |y_1| \leqslant 1\}$, while $G_0^h = G_0 \times (-{}^1/_2h, {}^1/_2h)$ is a plate of small relative thickness h (all coordinates are dimensionless), consisting of an elastic material with Lamé constants λ and μ. By d we denote an even smooth non-negative function on $[-{}^1/_2, {}^1/_2]$ and introduce a set $M^h = \{x \equiv (y, z) \in \mathbf{R}^3\colon y_2 = 0, |z| \leqslant {}^1/_2h, |y_1| \leqslant 1 + hd(h^{-1}z)\}$ which is a crack in G_0^h, the function d describing the shape of the end zones of the crack. We denote the cracked plate by $G^h = G_0^h \setminus M^h$ and assume that there are no body forces, the upper and lower faces $\Sigma_\pm^h$ of the plate and the edges $M_\pm^h$ of the crack are stress-free, while strain is induced by a self-balancing loading p applied to the edge surface Γ^h of the plate and having the form $p(x) = (p'(y), 0)$, where $p' = (p_1, p_2)$. The displacement vector $u = (u_1, u_2, u_3)$ is the solution of the boundary-value problem

$$L(\nabla_x)\, u(x) \equiv \mu\nabla_x\cdot\nabla_x u(x) + (\lambda+\mu)\nabla_x\nabla_x\cdot u(x) = 0, \quad x \in G^h \tag{1.1}$$

$$\sigma^{(3)}(u; x) = 0, \quad x \in \Sigma_\pm^h \tag{1.2}$$

$$\sigma^{(2)}(u; x) = 0, \quad x \in M_\pm^h; \quad \sigma^{(n)}(u; x) = p(x), \quad x \in \Gamma^h \tag{1.3}$$

$$(\nabla_x = \operatorname{grad}, \quad \sigma^{(j)} = \sigma e^{(j)}, \quad \sigma^{(n)} = \sigma n, \quad n = (n', 0))$$

The dot denotes the scalar product, $\sigma(u)$ is the three-dimensional stress tensor, $e^{(j)}$ is the unit vector in $\mathbf{R}^3$, and n' is the external unit normal vector to the boundary of the domain $G_0 \subset \mathbf{R}^2$.

It is well-known that the solution u of problem (1.1)-(1.3) can be approximated to some degree of accuracy by the solution $v^0 = (v_1^0, v_2^0)$ of the generalized plane-stress state problem

$$L'(\nabla_y)\, v^0(y) \equiv \mu\nabla_y\cdot\nabla_y v^0(y) + (\lambda'+\mu)\nabla_y\nabla_y\cdot v^0(y) = 0, \quad y \in G = G_0 \setminus M \tag{1.4}$$

$$\tau^{(2)}(v^0; y) = 0, \quad y \in M_\pm; \quad \tau^{(n)}(v^0; y) = p'(y), \quad y \in \Gamma \tag{1.5}$$

$$(\nabla_y = (\partial/\partial y_1, \ \partial/\partial y_2), \quad \lambda' = 2\lambda\mu(\lambda+2\mu)^{-1}, \quad \tau^{(n)} = \tau n', \quad \tau^{(2)} = (\tau_{12}, \ \tau_{22}))$$

Here $\tau(v^0)$ is the two-dimensional stress tensor (with constants λ' and μ).

In this section we shall only formulate an exact error estimate for the solution u of the problem of the stress-strain state of the plate G_0^h without the crack (the load being as before). We put $V^0(x, h) = D(h, z, \nabla_y)\, v^0(y)$, where D is a (3x2)-matrix differential operator defined by

$$D(h, z, \nabla_y)\, v = (v, 0) - \lambda[\lambda+2\mu]^{-1} z e^{(3)}\nabla_y\cdot v + {}^1/_2\lambda[\lambda+2\mu]^{-1}[z^2 - {}^1/_{12}h^2](\nabla_y\nabla_y\cdot v, 0) \tag{1.6}$$

The components of the corresponding three-dimensional stress tensor are found from the formulae

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$$\sigma_{jk}(V^0) = \tau_{jk}(v^0) + {}^1/_2\lambda(\lambda + 2\mu)^{-1}(z^2 - {}^1/_{12}h^2)\,\tau_{jk}(\nabla_y\nabla_y\cdot v^0) \tag{1.7}$$
$$\sigma_{j3}(V^0) = 0 \quad (j, k = 1, 2) \quad \sigma_{33}(V^0) = 0$$

We assume that $p_1, p_2 \in W_2^{5/2}(\Gamma)$. Then the solution v^0 of problem (1.4), (1.5) in the domain G_0 (without the crack M the first boundary condition in (1.5) is absent) belongs to $W_2^4(G_0)$. Using the results of /1-3/ we obtain the inequality

$$\| u_j - V_j^0 \| + h^{-1} \| u_3 - V_3^0 \| + \| (d + h)\,\sigma(u - V^0) \| \leqslant ch^{1/2} \tag{1.8}$$

Here $\|\cdot\|$ is the norm in $L_2(G_0^h)$, $d(x) = \operatorname{dist}(y, \Gamma)$ and the constant does not depend on h. Furthermore, it is assumed that the vectors u and V^0 are normalized by identical conditions which remove the arbitrariness in the choice of rigid displacement. (In particular, u_3 is an odd function of x_3). As in /3/, by requiring extra smoothness of the vector p' one can obtain estimates for the difference $u - V^0$ in weighted Hölder classes.

The solution v^0 of problem (1.4), (1.5) in the domain with a crack has an expansion

$$v^0(x) = c^0 + r^{1/2}(K_1\Phi^1(\varphi) + K_2\Phi^2(\varphi)) + (b_1^0x_1 + b_2^0x_2, -b_2^0x_1 - b_1^0\lambda'[\lambda' + 2\mu]^{-1}x_2) + r^{3/2}(k_1\Upsilon^1(\varphi) + k_2\Upsilon^2(\varphi)) + O(r^2) \quad (r \to 0) \tag{1.9}$$

$$(\Phi_r^1(\varphi), \Phi_\varphi^1(\varphi)) = (4\mu)^{-1}(2\pi)^{-1/2}([2\varkappa' - 1]\cos{}^1/_2\varphi - \cos{}^3/_2\varphi,\ \sin{}^3/_2\varphi - [2\varkappa' + 1]\sin{}^1/_2\varphi), \quad (\Phi_r^2(\varphi), \Phi_\varphi^2(\varphi)) = (4\mu)^{-1}(2\pi)^{-1/2}(3\sin{}^3/_2\varphi - [2\varkappa' - 1]\sin{}^1/_2\varphi,\ 3\cos{}^3/_2\varphi - [2\varkappa' + 1]\cos{}^1/_2\varphi) \tag{1.10}$$

$$(\Upsilon_r^1(\varphi), \Upsilon_\varphi^1(\varphi)) = (12\mu)^{-1}(2\pi)^{-1/2}(\cos{}^5/_2\varphi + [2\varkappa' - 3]\cos{}^1/_2\varphi,\ -\sin{}^5/_2\varphi + [2\varkappa' + 3]\sin{}^1/_2\varphi), \quad (\Upsilon_r^2(\varphi), \Upsilon_\varphi^2(\varphi)) = (12\mu)^{-1}(2\pi)^{-1/2}\cdot(5\sin{}^5/_2\varphi + [2\varkappa' - 3]\sin{}^1/_2\varphi,\ 5\cos{}^5/_2\varphi - [2\varkappa' + 3]\cos{}^1/_2\varphi)$$

$$(c^0 = (c_1^0, c_2^0), \ \varkappa' = (\lambda' + 3\mu)(\lambda' + \mu)^{-1} = (5\lambda + 6\mu)(3\lambda + 2\mu)^{-1})$$

Here c_j^0 and b_j^0 are certain constants, (r, φ) are polar coordinates centred on $(1, 0)$, $\varphi \in (-\pi, \pi)$; K_1 and K_2 are stress intensity coefficients (SICs), and k_1 and k_2 are the coefficients of the non-leading singular terms. There is a similar representation in the neighbourhood of the other vertex. (From now on we will use the notation $(r_\pm, \varphi_\pm)$, $K_{j\pm}$ etc.) We note that for the SICs we have the formulae /4, 5/

$$K_j = \int_\Gamma p'(y)\cdot\zeta^j(y)\,ds_y \quad (j = 1, 2) \tag{1.11}$$

Here the ζ^j are weight functions, solutions of the homogeneous $(p' = 0)$ problem (1.4), (1.5), bounded everywhere except at the point $y = (1, 0)$, and with asymptotic expansions

$$\zeta^j(x) = r^{-1/2}\Psi^j(\varphi) + \sum_{k=1,2} \mathrm{C}_{jk}r^{1/2}\Phi^k(\varphi) + O(r) \quad (r \to 0) \tag{1.12}$$

$$(\Psi_r^1(\varphi), \Psi_\varphi^1(\varphi)) = [\varkappa' + 1]^{-1}(8\pi)^{-1/2}(3\cos{}^1/_2\varphi - [2\varkappa' + 1]\cos{}^3/_2\varphi,\ -3\sin{}^1/_2\varphi + [2\varkappa' + 1]\sin{}^3/_2\varphi), \quad (\Psi_r^2(\varphi), \Psi_\varphi^2(\varphi)) = [\varkappa' + 1]^{-1}(8\pi)^{-1/2}(-\sin{}^1/_2\varphi + [2\varkappa' + 1]\sin{}^3/_2\varphi,\ -\cos{}^1/_2\varphi + [2\varkappa' - 1]\cos{}^3/_2\varphi) \tag{1.13}$$

The matrix C, composed of the factors C_{jk} in the expansions (1.12), depends on G, λ and μ and is symmetric. (For a boundary crack this matrix is positive-definite /6/, but here this property could be absent).

Returning to the discussion of the behaviour of the solution of problem (1.1)-(1.3) as $h \to 0$, we emphasise that because of the $O(r^{1/2})$ singularities in representations (1.9) the quantities (1.6) cannot serve as asymptotic approximations: the functions (1.7) are not, in general, square-integrable in G^h. The aim of this paper is to construct a global asymptotic expansion of the solution; as is usual in similar situations, it is necessary to investigate a boundary layer.

2. ***Model boundary layer problem.*** We denote by Π^0 the layer $\{\eta \in \mathbf{R}^3\colon |\eta_3| < {}^1/_2\}$, and by N the set $\{\eta : \eta_2 = 0,\ |\eta_3| \leqslant {}^1/_2,\ \eta_1 \leqslant d(\eta_3)\}$. The domain $\Pi = \Pi^0 \setminus N$ is obtained from the cracked plate G^h if we introduced "stretched" variables $\eta = (h^{-1}(y_1 - 1), h^{-1}y_2, h^{-1}z)$ and formally go to $h = 0$. The boundary layer that appears near the ends of the cut M^h is described with the help of special solutions U^j of the homogeneous elasticity theory problem

$$L(\nabla_\eta)W^j(\eta) = 0, \quad \eta \in \Pi; \quad \sigma^{(2)}(W^j; \eta) = 0, \quad \eta \in N_\pm \tag{2.1}$$
$$\sigma^{(3)}(W^j; \eta) = 0, \quad \eta \in \Pi_\pm = \{\eta : \eta_3 = \pm{}^1/_2\} \setminus N$$

This boundary layer has the form $h^{1/2}(K_1W^1(\eta)+K_2W^2(\eta))$, while from (1.9) and the matching conditions /7/ the solutions W^j are governed by the relations

$$W_k^j(\eta)=r_\eta^{1/2}\Phi_k^j(\varphi_\eta)+o(r_\eta^{1/2}),\quad k=1,2,\quad W_3^j(\eta)=o(r_\eta^{1/2})\quad (r_\eta\to\infty) \tag{2.2}$$

Here (r_η,φ_η) are polar coordinates corresponding to the two-dimensional Cartesian coordinates $\eta'=(\eta_1,\eta_2)$. Using the symmetry of the domain Π with respect to the $\eta_20\eta_3$ plane, we obtain, in accordance with /8/, the more-accurate representation

$$W^j(\eta)=D(1,\eta_3,\partial/\partial\eta_1,\partial/\partial\eta_2)\{r_\eta^{1/2}\Phi^j(\varphi_\eta)+c_j(d)\,r_\eta^{-1/2}\Psi^j(\varphi_\eta)\}+O(r_\eta^{-1})\quad (r_\eta\to\infty) \tag{2.3}$$

In (2.3) D is the operator in (1.6), the $c_j(d)$ are quantities depending on μ,λ and the shape of the end of the cut N (the function d).

Let (ρ_η,θ_η) be polar coordinates in the plane through the point $\eta=(d(\eta_3),0,\eta_3)$ and perpendicular to the arc $T=\{\eta:\eta_2=0,\ |\eta_3|<1/2,\ \eta_1=d(\eta_3)\}$, with $|\theta_\eta|<\pi$ and $|\eta_3|<1/2$. Near the face T of the crack N the solutions of problems (2.1), (2.2) can be expanded as

$$\begin{gathered}W^1(\eta)=\beta_1(\eta_3)e^{(1)}+\beta_3(\eta_3)e^{(3)}+t_1(\eta_3)\rho_\eta^{1/2}\Phi_*^1(\theta_\eta)+O(\rho_\eta(\rho_\eta-\\ |\eta_3|+1/2)^{\operatorname{Re}\Lambda-1-\delta}),\quad W^2(\eta)=\beta_2(\eta_3)e^{(2)}+t_2(\eta_3)\rho_\eta^{1/2}\Phi_*^2(\theta_\eta)+\\ t_3(\eta_3)\rho_\eta^{1/2}\Phi_*^3(\theta_\eta)+O(\rho_\eta(\rho_\eta-|\eta_3|+1/2)^{\operatorname{Re}\Lambda-1-\delta})\quad(\rho_\eta\to0,\\ |\eta_3|<1/2)\end{gathered} \tag{2.4}$$

Here $e^{(m)}$ is a unit vector in $\mathbf{R}^3$, $\Phi_*^j=(\Phi_{*1}^j,\Phi_{*2}^j,0)$, $j=1,2$ are the angular dependences given by formulae (1.10) with $\varkappa'$ replaced by $\varkappa=(\lambda+3\mu)(\lambda+\mu)^{-1}$; $\Phi_*^3(\theta)=(0,0,\mu^{-1}(2\pi)^{-1/2}\sin{}^1/_2\theta)$; β_m and t_m are smooth functions on $(-1/2,1/2)$, and δ is an arbitrary positive number. The residue estimate in (2.4) and the behaviour of the functions β_m, t_m as $\eta_3\to\pm1/2$ are governed by the fact that the points $P_\pm=(d(1/2),0,\pm1/2)$ are polyhedron vertex type singularities for the boundary $\partial\Pi$. $\operatorname{Re}\Lambda$ correspondingly denotes the smallest positive real part of the degree of homogeneity (in terms of distance from P_+) of the solution of the three-dimensional elasticity theory problem in the domain $\Xi=\{x\in\mathbf{R}^3:\ x_3<1/2\}\setminus\{x:\ x_1\leqslant d'(1/2)(x_3-1/2,\ x_2=0)\}$, in the half-space with an angularly-shaped crack. Such indices $\Lambda(\alpha,\nu)$ were computed in /9, 10/. They depend on the angle $\alpha=\operatorname{arctg}d'(1/2)+1/2\pi$ of the crack aperture and on Poisson's ratio $\nu=1/4(3-\varkappa)$, with $\operatorname{Re}\Lambda(\alpha,\nu)\geqslant0$, while the function $\alpha\to\Lambda(\alpha,\nu)$ is continuous in $[0,\pi]$ and $\Lambda(0,\nu)=0$, $\Lambda(\pi,\nu)=1$ (see e.g. /11/). Because the asymptotic behaviour of the vector function U^j for $\eta\to P_\pm$ is governed by the indices $\Lambda(\alpha,\nu)$ (or more precisely, the cited solutions of the three-dimensional problem), the derivatives of the functions β_m and t_m in general become infinite at $\eta_3=\pm1/2$.

The estimates

$$\begin{gathered}|\beta_m(\eta_3)|\leqslant c_0,\ |[(1/4-\eta_3^2)\partial_3]^{n+1}\beta_m(\eta_3)|\leqslant c_{n+1}(1/4-\eta_3^2)^{\operatorname{Re}\Lambda-\delta}\\ |[(1/4-\eta_3^2)\partial_3]^n t_m(\eta_3)|\leqslant c_n(1/4-\eta_3^2)^{\operatorname{Re}\Lambda-1/2-\delta}\\ m=1,2,3,\ n=0,1,\ldots,\ \partial_m=\partial/\partial\eta_m,\ \delta>0\end{gathered} \tag{2.5}$$

will be used below.

We now consider the differences in the powers of the quantities $1/4-\eta_3^2$, being the distances in T from the points $P_\pm$. In the first of the estimates (2.5) account is taken of the fact that there are rigid translational displacements in the asymptotic expansion of W^j near $P_\pm$ and it is they which govern the behaviour of $\beta_m(\eta_3)$ as $\eta_3\to\pm1/2$. The ∂_3 differentiation eliminates these terms from the expansion and the leading term becomes the derivative of the homogeneous solution in the polyhedron Ξ, which in spherical coordinates $(\rho_+,\theta_+,\varphi_+)$ with centre P_+ has the form

$$c_+\rho_+^{\Lambda(\alpha,\nu)}\Upsilon(\theta_+,\varphi_+),\quad c_+=\text{const} \tag{2.6}$$

The angular part Υ is the solution of a boundary-value problem in the domain Ω_α, cut out of the unit sphere S^2 by the polyhedron Θ. The domain Ω is a hemisphere with a distant arc of a great circle, and three angular points are situated on its boundary $\partial\Omega_\alpha$: one of the points (denoted by Q) is the vertex of a one-dimensional crack, while smooth arcs of the contour $\partial\Omega_\alpha$ meet at a right at the other two points. One can verify that the asymptotic expansion of the vector function Υ in a small neighbourhood of the point Q, to an accuracy of $O(d)$, where d is the distance on the sphere from Q, is identical with the similar asymptotic expansion near the vertex of the crack in the plane and antiplane problems of elasticity theory. Hence solution (2.6) contributes to each term of the expansion (2.4) near the edge T. In other words, we have the relation (given certain additional conditions)

$$\beta_m(\eta_3)=\beta^0_{m\pm}+c_\pm\beta^\Lambda_{m\pm}(1/4-\eta_3^2)^\Lambda+o((1/4-\eta_3^2)^\Lambda)\quad(\eta_3\to\pm 1/2)$$

This relation also reduces to the second of the estimates (2.5). The halving of the power in the estimates for t_m is associated with the fact that $\rho_+\sim\rho_+d$ and means that the $O(\rho_+^{1/2})$ part of the $O(\rho_+^\Lambda)$ singularity of function (2.6) at the edge is already accounted for in formula (2.4) itself. Finally, the angular part of (2.6) depends polynomially on $\ln\rho_+$ when the spectral problem in Ω_α has an adjoint vector, hence an arbitrary positive number δ is introduced into (2.4) and (2.5).

The mathematical jusitification of these facts is given in /12/.

We will derive an identity for the function k_m. With this aim we note that the derivative $\partial_1W^j\equiv\partial W^j/\partial\eta_1$ satisfies equalities (2.1), while relation (2.2) takes the following form (compare with the discussions in /6/):

$$\partial_1W^j(\eta)=-(4\mu)^{-1}(1+\varkappa')D(1,\eta_3,\partial/\partial\eta_1,\partial/\partial\eta_2)r_\eta^{-1/2}\Psi^j(\varphi_\eta)+O(r_\eta^{-1})\quad(r_\eta\to\infty)\tag{2.7}$$

Differentiating the expansions (2.4) with respect to η_1, we have

$$\begin{aligned}\partial_1W^1(\eta)&\sim-(4\mu)^{-1}(1+\varkappa)[1+d'(\eta_3)^2]\,t_1(\eta_3)\rho_\eta^{-1/2}\Psi_*^1(\theta_\eta)\\ \partial_1W^2(\eta)&\sim-(4\mu)^{-1}(1+\varkappa)[1+d'(\eta_3)^2]\,\{t_2(\eta_3)\rho_\eta^{-1/2}\Psi_*^2(\theta_\eta)+\\&\quad 4(1+\varkappa)^{-1}t_3(\eta_3)\rho_\eta^{-1/2}\Psi_*^3(\theta_\eta)\}\quad(\rho_\eta\to 0)\end{aligned}\tag{2.8}$$

Here the angular parts Ψ_*^1 and Ψ_*^2 are determined by (1.13) just like Φ_*^1 and Φ_*^2 were determined by (1.10), and $\Psi_*^3(\theta)=(0,0,2(2\pi)^{-1/2}\sin{}^1/_2\theta)$. We will not give estimates for the errors in (2.8), but they can be obtained from (2.4) and (2.5).

We denote by T_ε the cylindrical ε-neighbourhood of the set T and apply the Betti formula in the domain $\{\eta\in\Pi\colon r_\eta<\varepsilon^{-1}\}\setminus\overline{T}_\varepsilon$ for the fields W^j and ∂_1W^j As a result we obtain an equality connecting the integrals of the difference $\sigma^{(n)}(W^j)\,\partial_1W^j-\sigma^{(n)}(\partial_1W^j)\,W^j$ over the surface $\{\eta\in\Pi\colon r_\eta=\varepsilon^{-1}\}$ and $\partial T_\varepsilon\cap\Pi$. We make ε vanish and to compute the limits of the integrals we use estimates (2.5) and the asymptotic expansions (2.2), (2.7) and (2.4), (2.8), respectively. Furthermore, we note that

$$\lim_{\varepsilon\to 0}\int_{\{\rho_\eta=\varepsilon\}}\{\sigma^{(\rho)}(\rho_\eta^{1/2}\Phi_*^j)\rho_\eta^{-1/2}\Psi_*^k-\sigma^{(\rho)}(\rho_\eta^{-1/2}\Psi_*^k)\rho_\eta^{1/2}\Phi_*^j\}\,ds_\theta=\delta_{j,k}$$

Changing to integrals of the first kind over T (with the factor $1+d'(\eta_3)^2$ in (2.8)), we finally obtain the identity

$$\frac{1+\varkappa'}{1+\varkappa}=\int_T t_1(\eta_3)^2\,ds_3=\int_T t_2(\theta_3)^2\,ds_3+\frac{4}{1+\varkappa}\int_T t_3(\eta_3)^2\,ds_3\tag{2.9}$$

3. ***The leading term of the global asymptotic expansion.*** From the expansion (1.9) of the smooth v^0 solution the matching procedure /7/ determines the behaviour at infinity of the $K_1W^1+K_2W^2$ type boundary layer solution (problem (2.1), (2.2)). In exactly the same way the matching of the representation (2.3) with the smooth $D(h,z,\nabla_y)(v^0+hv^1+\ldots)$ solution determines the asymptotic formula (as $r\to 0$) for v^1. Specifically, the vector function v^1 should be governed by the relations

$$L'(\nabla_y)\,v^1(y)=0,\quad y\in G_0\setminus M;\quad \tau^{(n)}(v^1;y)=0,\quad y\in\Gamma\cup M\tag{3.1}$$

$$v^1(y)=\sum_{j=1,2}c_j(d)K_{j\pm}r_\pm^{-1/2}\Psi^j(\varphi_\pm)+c_\pm^1+O(r_\pm^{1/2})\quad(r_\pm\to 0)\tag{3.2}$$

The plus and minus signs correspond to the $(1,0)$ and $(-1,0)$ vertices of the crack M. In accordance with the definition of the weight functions $\zeta^{1\pm}$ and $\zeta^{2\pm}$ from (3.2) and (1.12) we derive

$$v^1(y)=\sum_\pm\sum_{j=1,2}c_j(d)K_{j\pm}\zeta^{j\pm}(y)\tag{3.3}$$

We shall formulate an estimate for the residue in the two-term asymptotic expansion obtained; the derivation of this estimate differs very little from /3/ - it is only necessary to take into account the appearance of additional singularities, which is done, in part, with the help of results from /13/. To simplify the notation we shall only deal with the $(1,0)$

vertex of the crack. We denote by χ a cut-off function in $C_0^\infty(\mathbf{R})$ such that $\chi(r)=0$ for $r>{}^1/_2$ and $\chi(r)=1$ for $r<{}^1/_4$, while the asymptotic approximation to the solution of problem (1.1)-(1.3) is determined by

$$V^1(x)=(1-\chi(h^{-1}r))D(h,z,\nabla_y)(v^0(y)+hv^1(y))+\chi(h^{-1}r)(c^0+hc^1,0)+\\ \chi(r)\sum_{j=1,2}K_j\{h^{1/2}W^j(\eta)-(1-\chi(h^{-1}r))D(h,z,\nabla_y)(r^{1/2}\Phi^j(\varphi)+\\ hc_j(d)r^{-1/2}\Psi^j(\varphi))\} \tag{3.4}$$

We stress that this complicated construction is only necessary to justify the asymptotic solution. Thanks to the representations (1.9), (1.12) and (2.3) one can see that $V^1\sim D(v^0+hv^1)$ far from the face zones of the crack and that $V^1\sim(c^0+hc^1,\ 0)+h^{1/2}(K_{1+}W^1+K_{2+}W^2)$ near its right-hand end. Taking the field (3.4) and the solution u of problem (1.1)-(1.3) to be governed by the same orthogonality conditions eliminating the arbitrariness in the choice of the rigid displacement, we obtain the estimate

$$\|u_1-V_1^1\|+\|u_2-V_2^1\|+h^{-1}\|u_3-V_3^1\|+\|\sigma(u-V^1)\|\leqslant ch^{3/2} \tag{3.5}$$

We note that the power of h in (3.5) is smaller than in (1.8). The point is that in the case of a smooth contour bounding the middle section of the plate, smooth solutions v^1 and v^2 vanish, i.e. the vector function v^0 itself generates a third-order approximation. It turns out that the perturbation arising from the singular point of the specified contour has order h. In other words, the asymptotic solution of the three-dimensional problem contains non-zero terms hv^1 and h^2v^2; the latter is included in the residue, which "coarsens" estimate (3.5) relative to (1.8). One could also have determined the v^2 term: the corresponding term $h^{3/2}W(\eta)$ of the boundary layer is obtained by including expressions of orders $r^{3/2}$ and $r^{1/2}$ in expansions (1.9) and (1.12), respectively.

We will give some corollaries to relations (3.4) and (3.5).

1) Suppose U^h and U_0 are the strain potential energies, and A^h and A_0 are the work of external forces for problems (1.1)-(1.3) and (1.4), (1.5) respectively. Because $V^1=D(v^0+hv^1)$ near Γ^h, then from (3.5), (1.7) and (3.3), (1.11) we derive the equalities

$$U^h=-{}^1/_2A^h=-{}^1/_2h\int_\Gamma p'(y)\cdot(v^0(y)+hv^1(y))\,ds_y+O(h^3)=\\ hU_0-{}^1/_2h^2\sum_\pm(c_1(d)K_{1\pm}^2+c_2(d)K_{2\pm}^2)+O(h^3) \tag{3.6}$$

2) Suppose Λ_0 is a simple eigenvalue of the operator of problem (1.4), (1.5) and v^0 is the corresponding eigenvector normalized in $L_2(G)$. Expansion (1.9) holds for v^0. It is known that the eigenfrequency $\omega_0(h)$ of longitudinal oscillations of a three-dimensional plate with material density γ satisfies the estimate $|\omega_0(h)-\Lambda_0^{1/2}\gamma^{-1/2}|\leqslant \mathrm{const}\,h$. Use of the second term of the asymptotic expansion gives a more precise formula for the eigenfrequency:

$$\omega_0(h)=\gamma^{-1/2}\left(\Lambda_0+h\sum_\pm\sum_{j=1,2}c_j(d)K_{j\pm}^2\right)^{1/2}+O(h^2)$$

3) Consider the problem of the quasistatic propagation of the cut M^h. We put $M(\varepsilon)=\{y\colon y_2=0,\ -1\leqslant y_1\leqslant 1+\varepsilon\}$, $M^h(\varepsilon)=\{x\colon y_2=0,\ |z|\leqslant{}^1/_2h,\ -1-hd(h^{-1}z)\leqslant y_1\leqslant 1+\varepsilon+hd(h^{-1}z)\}$, where $0<\varepsilon$ is a positive parameter, (i.e. the crack grows on one side and the shape of the end zone remains unchanged). We denote by $U^h(\varepsilon)$ the strain potential energy of the plate G_0^h with crack $M^h(\varepsilon)$, and by $U_0(\varepsilon)$ the potential energy corresponding to the plane-stress state of the domain G_0 with crack $M(\varepsilon)$, while $K_j(\varepsilon)$ are the corresponding CSIs.

Near the edge $T_+^h=\{x\colon y_2=0,\ |z|<{}^1/_2h,\ y_1=1+hd(h^{-1}z)\}$ the solution u of problem (1.1)-(1.3) has the representation

$$u(x)=\sum_{m=1}^3\{\mathbf{B}_m(z)e^{(m)}+\mathbf{K}_m(z)\rho^{1/2}\Phi_*^m(\theta)\}+\\ O(\rho(\rho-|z|+{}^1/_2h)^{\mathrm{Re}\,\Lambda-1-\delta})\quad(\rho\to 0) \tag{3.7}$$

similar to (2.4). Here ρ,θ are polar coordinates in the plane crossing the point $(1+hd(h^{-1}z),\ 0,\ z)$ where $|z|<{}^1/_2h$, and perpendicular to the arc T_+^h, while $\mathbf{K}_m(z)$ is the CSI. Using relations (3.4) and (3.5) one can verify that the estimates

$$|\mathbf{K}_1(z)-t_1(h^{-1}z)K_1|+|\mathbf{K}_p(z)-t_p(h^{-1}z)K_2|\leqslant \mathrm{const}\,h\,({}^1/_4h^2-\\ z^2)^{\mathrm{Re}\,\Lambda-1/2-\delta}\quad(p=2,3) \tag{3.8}$$

hold (see (2.5)).

Now, using (2.9) and the well-known Griffith-Irwin formula, we find

$$dU^h/d\varepsilon\,|_{\varepsilon=0} = -(2\mu)^{-1}\int_{T_+^h}\{{}^1/_4(1+\varkappa)(\mathbf{K}_1(z)^2+\mathbf{K}_2(z)^2)+\mathbf{K}_3(z)^2\}\,ds_x = \tag{3.9}$$

$$-(2\mu)^{-1}h\int_T\{{}^1/_4(1+\varkappa)(K_1{}^2t_1(\eta_3)^2+K_2{}^2t_2(\eta_3)^2)+K_2{}^2t_3(\eta_3)^2\}\,ds_\eta +$$

$$O(h^2) = -(8\mu)^{-1}h(1+\varkappa')(K_1{}^2+K_2{}^2)+O(h^2)$$

The rates of energy release computed according to the three-dimensional and two-dimensional Griffith-Irwin formulae are consequently identical to within $O(h^2)$.

To improve the residue estimate we apply relation (3.6). We have

$$\left.\frac{dU^h}{d\varepsilon}\right|_{\varepsilon=0} = h\left.\frac{dU_0}{d\varepsilon}\right|_{\varepsilon=0} - h^2\sum_{\pm}\sum_{j=1,2}c_j(d)K_{j\pm}(0)\frac{dK_{j\pm}}{d\varepsilon}(0)+O(h^3)$$

The CSI derivatives along the length of the crack are determined in /6, 14/. We recall and introduce some notation: $K_{j\pm}\equiv K_{j\pm}(0)$ and $k_{j\pm}$ are the coefficients in expansion (1.9) of the vector v^0 near the vertices $(\pm 1, 0)$ of the crack M; the $\mathbf{C}_{jk}{}^+$ are the factors $\mathbf{C}_{jk}$ in representation (1.12) of the special solutions ζ^{j+} of problem (3.1); these solutions have singularities of $O(r^{-1/2})$ at the point $(1, 0)$ and are bounded in a neighbourhood of the other singular point $(-1, 0)$, while the coefficients K_q, $q = 1, 2$ in expansions of the form (1.9) of the vectors ζ^{j+} for $r_-\to 0$ are the $\mathbf{C}_{jq}{}^-$. We have the relations

$$K_{j+}(\varepsilon) = K_{j+}+\varepsilon\{{}^1/_2k_{j+}+(4\mu)^{-1}(1+\varkappa')\sum_{p=1,2}K_{p+}\mathbf{C}^+_{pj}\}+O(\varepsilon^2) \tag{3.10}$$

$$K_{j-}(\varepsilon) = K_{j-}+\varepsilon(4\mu)^{-1}(1+\varkappa')\sum_{p=1,2}K_{p+}\mathbf{C}^-_{pj}+O(\varepsilon^2)$$

Thus

$$\left.\frac{dU^h}{d\varepsilon}\right|_{\varepsilon=0} = -(8\mu)^{-1}(1+\varkappa')h\sum_{j=1,2}K^2_{j+} - h^2\sum_{j=1,2}c_j(d)\times \tag{3.11}$$

$$\Big\{K_{j+}\Big[{}^1/_2k_{j+}+(4\mu)^{-1}(1+\varkappa')\sum_{p=1,2}K_{p+}\mathbf{C}^+_{pj}\Big]+(4\mu)^{-1}(1+\varkappa')K_{j-}\times$$

$$\sum_{p=1,2}K_{p+}\mathbf{C}^-_{pj}\Big\}+O(h^3) = -h(8\mu)^{-1}(1+\varkappa')\times$$

$$\sum_{j=1,2}\{K_{j+}+hc_j(d)[(2\mu)^{-1}(1+\varkappa')k_{j+}+K_{1+}\mathbf{C}^+_{1j}+K_{2+}\mathbf{C}^+_{2j}]+$$

$$hc_1(d)K_{1-}\mathbf{C}^-_{j1}+hc_2(d)K_{2-}\mathbf{C}^-_{j2}\}^2+O(h^3)$$

Comparing (3.9) with (3.11), the latter ends with terms of $O(h^2)$ which come from the asymptotic expansion of the CSI $\mathbf{K}_m(z)$ as $h\to 0$ (see (3.7) and (3.8)). Exactly the same formula is obtained if one constructs an $h^{3/2}W(\eta)$ term for the boundary layer mentioned after estimate (3.5) and uses transformations similar to (3.9).

4. The "unified problem" for two-term asymptotic forms. As shown by examples 1)-3) of Sect.3, when computing various integral characteristics of the stress-strain state of a thin cracked plate the boundary layer only plays an intermediary role: the correction term in the asymptotic expansions of such characteristics is determined by the second term (3.3) of the smooth expansion, which contains only the quantities $c_1(d)$ and $c_2(d)$ from representation (2.3) of the special boundary-layer solutions W^1 and W^2. Hence it is reasonable to formulate problems in the domain $G = G_0\setminus M\subset\mathbf{R}^2$ whose solutions are identical to $O(h^2)$ with the sum v^0+hv^1, where because of the neglect of the three-dimensional boundary layers it is sufficient to require that the solutions be identical outside small neighbourhoods of the vertices of the crack M.

1) As is usual in the method of matched asymptotic expansions, the non-leading terms of the asymptotic expansion possess non-energy singularities at the irregular boundary points (compare (3.3) and (1.12)). Hence one of the possible statements of the problem uniting two asymptotic terms can be the widening of the domain of definition of the operator of problem (1.4) and (1.5). Concepts of this kind have been in use for a long time, for example, in diffraction problems (the theory of zero radius potentials, see /15, 16/ and others). We

will quote the results applicable to problem (1.4) and (1.5) with mass forces f:

$$-L'(\nabla_y)\, v(y) = f(y),\ y \in G;\quad \tau^{(n)}(v; y) = 0,\ y \in \partial G \tag{4.1}$$

Let $\mathbf{L}$ be an (unbounded) operator in $L_2(G)$, presecribed by a differential expression $L'(\nabla_y)$ and having a domain of definition $D(\mathbf{L}) = \{v \in W_2^2(G)\colon v(\pm 1, 0) = 0$ and $\tau^{(n)}(v) = 0$ on $\partial G\}$; this operator is closed and symmetric. Suppose also that $\mathbf{L}^e$ is the operator given by the same differential expression, but with the following domain of definition:

$$D(\mathbf{L}^e) = \{v : v(y) = w(y) + \sum_{\pm}[\chi(r_\pm) c^{0,\pm} + \sum_{j=1,2} K_{j\pm}(hc_j(d)\zeta^{j\pm}(y) + \chi(r_\pm) r_\pm^{1/2}\Phi^j(r_\pm))],\ w \in D(\mathbf{L}),\ c^{0\pm} \in \mathbf{R}^2,\ K_{j\pm} \in \mathbf{R}\} \tag{4.2}$$

One can verify that the operator $\mathbf{L}^e$ is a selfconjugate extension of the operator $\mathbf{L}$, and that the solution v^e of the equation $-\mathbf{L}^e v^e = f \in L_2(G)$ is equal to the sum $v^0 + hv^1$, in which $v^0 \in W_2^1(G)$ solves problem (4.1), and v^1 is the vector (3.3). The basic property that allows one to give a physical interpretation of the displacement field v^e as a "far field" is that the quadratic form $1/_2 h \langle \mathbf{L}^e v, v\rangle$, computed for the solution v^e of the above equation, is identical with the first two terms of the asymptotic expansion (3.6) of the potential energy U^h in the original three-dimensional problem. (This is verified with the help of formulae (4.2) and (1.11)).

2) Consider the problem of a normal separation crack, i.e. assume that the data of problem (1.1)-(1.3) are symmetric with respect to the $x_2 0 x_3$ plane. Then $K_2 = 0$ and only the quantity $c_1(d)$ will appear in formula (3.3). This enables us to introduce a unified problem by methods different from 1).

We first consider the couple stress theory of elasticity with constrained rotation

$$\begin{gathered}\mu\nabla_y\nabla_y w(y) + (\mu + \lambda')\nabla_y\nabla_y \cdot w(y) - 4\mu l^2\Theta(\nabla_y)\nabla_y\nabla_y\omega(y) = 0\\ \omega(y) = \Theta(\nabla_y)\, w(y),\ y \in G\\ \tau^{(n)}(w; y) - 4\mu l^2\Theta(n'(y))\nabla_y \cdot \nabla_y\omega(y) = p'(y),\ n'(y)\nabla_y\omega(y) = 0,\quad y \in \partial G\\ (\Theta(\nabla_y) = -1/_2(\partial/\partial y_2,\ -\partial/\partial y_1),\quad \Theta(n') = -1/_2(n_2,\ -n_1))\end{gathered} \tag{4.3}$$

Here $w = (w_1, w_2)$ is the displacement vector, ω is the rotation, n' is the unit external normal vector, $\tau(w)$ is the same (classical) stress tensor as in (1.5), and l, the couple index, is a small parameter.

We shall interpret problem (4.3) as a regularly degenerate problem as $l \to 0$ with a small parameter for the higher derivatives and use a modification /17/ of the Vishik-Lyusternik method /18/. Problem (1.4), (1.5) is a limiting case for (4.3). In the construction of the asymptotic expansion two boundary layers appear for the smooth boundary and for the angular point. The contribution of the first of these to the asymptotic displacement w is /18/ of $O(l^2)$. The angular boundary layer is determined /17, 19/ by the solution of the problem

$$\begin{gathered}\mu\nabla_\xi\nabla_\xi Z(\xi) + (\mu + \lambda')\nabla_\xi\nabla_\xi Z(\xi) - 4\mu\Theta(\nabla_\xi)\nabla_\xi\nabla_\xi\Omega(\xi) = 0\\ \Omega(\xi) = \Theta(\xi)\, Z(\xi),\quad \xi \in \mathbf{R}^2 \setminus N^0\ (N^0 = \{\xi : \xi_2 = 0,\ \xi_1 \leqslant 0\})\\ 2\mu\partial_2 Z_2(\xi) + \lambda'\nabla_\xi Z(\xi) = 0,\quad \mu(\partial_2 Z_1(\xi) + \partial_1 Z_2(\xi)) - 2\mu\nabla_\xi\nabla_\xi\Omega(\xi) =\\ 0,\quad \partial_2\Omega(\xi) = 0,\quad \xi \in N^0\quad (\partial_k = \partial/\partial\xi_k)\\ Z(\xi) = r_\xi^{1/2}\Phi^1(\theta_\xi) + O(r_\xi^{-1/2}),\quad \Omega(\xi) = O(r_\xi^{-1/2})\quad (r_\xi \to 0)\end{gathered}$$

We will make the behaviour of the vector Z at infinity more precise.

$$Z(\xi) = r_\xi^{1/2}\Phi^1(\theta_\xi) + mr_\xi^{-1/2}\Psi^1(\theta_\xi) + o(r_\xi^{-1/2}).$$

The factor m in the latter formula depends on the Lamé parameters λ' and μ. We match the boundary layers $l^{1/2}Z(-l^{-1}(1 \mp y_1), \pm l^{-1}y_2)$ with a smooth solution; we obtain the result that the second term in the smooth solution is the sum $lmK_{1+}\zeta^{1+} + lmK_{2+}\zeta^{2+}$, exactly similar to (3.3). Thus, if the parameter $l = hc_1(d)/m$ (small for $h \to 0$) is positive (this depends on the sign of $c_1(d)$), then the displacement vector w from problem (4.3) is different from the asymptotic approximation $v^0 + hv^1$ constructed in Sect.3 by a quantity of $O(h^2)$ outside small neighbourhoods of the points $(\pm 1, 0)$. In other words, for a special choice of the index l the solution of the plane couple stress problem (4.3) far from the vertices of the crack contains the next term of the asymptotic solution of the three-dimensional problem.

3) Another way of taking account of asymptotic corrections for normal separation cracks consists of changing the length of the "plane projection" M of the crack M^h. We put $M_\varepsilon = \{y \in \mathbf{R}^2\colon y_2 = 0,\ |y_1| \leqslant 1 + \varepsilon\}$, where $\mathbf{R} \ni \varepsilon$ is a small parameter, and consider problem (1.4),

(1.5) in the domain $G_0 \setminus M_\varepsilon$. The asymptotic solution of such a problem is found, for example, in /6/, and far from the points $(\pm 1 \pm \varepsilon, 0)$ has the form $v^0 + \varepsilon z^1 + \ldots$, with $z^1 = (4\mu)^{-1}(1+\varkappa')(K_{1+}\zeta^{1+} + K_{2+}\zeta^{2+})$. Comparison with formula (3.3) gives the value $\varepsilon_* = 4\mu h(1+\varkappa')^{-1} c_1(d)$ for the parameter ε, for which the solution of problem (1.4), (1.5) in $G_0 \setminus M_\varepsilon$ is identical to $O(h^2)$ outside small neighbourhoods of the crack vertices with the smooth solution $v^0 + hv^1 + \ldots$ occurring in the representation of the solution of problem (1.1)-(1.3).

We emphasise that here, unlike in the case of the problem in 2) of Sect.4, there is no restriction on the sign of $c_1(d)$ - the parameter ε can be negative. The length $2(1+\varepsilon_*)$ will be called the (asymptotic) effective length of the crack M^h.

5. *Discussion.* 1) We turn our attention to the fact that for a thin cracked plate the plane-stress state approximation gives an error of $O(h)$ and not $O(h^3)$, as in the case of a smooth guiding cylinder $G_0{}^h$. Specifically, the three-dimensional boundary layer for the end zones of the crack is not localized in small neighbourhoods and introduces an $O(h)$ perturbation into the entire stress-strain state of the plate. Within the framework of brittle failure such a boundary layer can give rise to additional effects "of order h" not included in classical theory. The description of the influence of the boundary layer globally throughout the plate has to be conducted one way or the other with the help of displacement fields with singularities at the vertices of the crack M (compare parts 1)-3) of Sect.4). The singularities are not significant in determining the energy balance in the correct interpretation of the energy functional (see part 1. of Sect.4): the verticres of the crack make additional contributions to the function on account of the $\zeta^{j\pm}$ terms.

Thus the application of energy criteria for failure is not made any harder in the more-accurate model. The use of strain, force and other "local" criteria would perhaps be impossible without exhaustive information about the behaviour of the stress-strain state near the ends of the crack. However, one finds that statements of problems with these criteria are asymptotically equivalent (for small connection zones at the crack aperture, for small plasticity zones, etc.) to selfconjugate extensions of the operator of problem (1.4), (1.5), similar to (4.2). (The corresponding asymptotic analysis has been performed by many authors, but not formulated in this way). Hence, after comparison of the parameters of the self-conjugate extension the above criteria also becomes suitable, for example, for a unified problem (part 3) of Sect.4).

2) When changing the profile d of the ends of the crack zone M^h the effective length $2a_*(d) = 2[1 + 4\mu h(1+\varkappa')^{-1} c_1(d)]$ of its plane projection may be increased by the growth of the factor $c_1(d)$, at the same time as its visible length $2[1 + hd(^1/_2)]$ (or $2[1 + h \max d(\eta_3)]$) remains constant. One can establish that comparison of the strain energies for the two problems (1.4) and (1.5) in the domain G_0 with cracks M_{ε^1} and M_{ε^2}, corresponding to different profiles d^1 and d^2, gives an approximation of order $O(h^3)$ to the energy change in the three-dimensional problem.

The following conjecture is natural: during quasistatic motion the crack "selects" a profile d such that the coefficient $\mathbf{K}_1(z)$ is constant for $z \in [-^1/_2 h, ^1/_2 h]$ (see expansion (3.7) where $\mathbf{K}_2 = \mathbf{K}_3 = 0$ for normal separation cracks). Estimates (3.8) show that the required profile is determined to some order of accuracy by solving the following problem: find a function $\mathbf{d}$ for which the quantity $t_1(\eta_3)$ from representation (2.4) of the solution W^1 of problem (2.1), (2.2) in the domain $\Pi = \Pi^0 \setminus \{\eta\colon \eta_2 = 0, |\eta_3| \leqslant ^1/_2, \eta_1 \leqslant d(\eta_3)\}$ is constant. There is a simple solution only in the case $\nu = 0$, when $d = 0$ and $W^1(\eta) = r_\eta^{1/2}\Phi_*^1(\theta_\eta)$. We remark that the condition $t_1 = \mathrm{const}$ imposes a restriction on the angle $\alpha = \operatorname{arctg} d'(^1/_2) + ^1/_2\pi$ - it should satisfy $\Lambda(\alpha, \nu) = ^1/_2$ (see the explanations for formula (2.5)).

The hypothetical process of quasistatic crack growth splits into two stages: the fracture first proceeds along some portions of the edge (establishing the necessary profile form) and only then does the growth of the crack reduce to parallel transport of the end zone. The first stage is brief, but can be associated with anomalous behaviour at the initiation of the crack.

3) Because the growth of the strain potential energy is expressed in terms of the CSI, it should be governed only by the behaviour of the stresses near the moving end of the crack. At first glance relation (3.11) contradicts this assertion - the coefficients K_{j-} for the fixed vertex of the crack M appear on the right-hand side of (3.11). The explanation is that the quantities K_{j+} and K_{j-} correspond only to the dominant term of the asymptotic expansion of the solution of the three-dimensional problem, while the following term hv^1 is constructed from formula (3.3), taking into account all the singularities of the field $\tau(v^0)$ and the terms in (3.11) containing K_{j-} appear because of the inclusion of the energy generated by the field $h\tau(v^1)$. Using equalities (3.10) one can verify that if the leading term in the asymptotic expansion is chosen to be the solution v^* of problem (1.4), (1.5) in a plane domain with a crack of effective length $2a_*(d)$, then the discrepancy vanishes: the derivative of the energy with respect to the crack length becomes equal to $-h(8\mu)^{-1}(1+\varkappa')(K_{1+}{}^*)^2$, where $K_{1+}{}^*$ is the CSI for the stresses $\tau(v^*)$.

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